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MATHEMATICAL PROPERTIES OF 9×9 STRONGLY MAGIC SQUARES

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ABSTRACT

A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discusses about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. This paper focuses on discovering the mathematical properties of 9×9 Strongly Magic Squares

KEYWORDS: Magic Square; Magic Constant;Strongly Magic Square .

AMS Classification Code:- 11XX,65XX,00AXX.

INTRODUCTION

Magic squares have turned up throughout history, some in a mathematical context and others in religious contexts. Magic squares date back to the first millennium B.C.E in China [1].It was a magic square of order three thought to have appeared on the back of a turtle emerging from a river. Other magic squares surfaced at various places around the world in the centuries following their discovery. Some of the more interesting examples were recorded in Europe during the 1500s. Cornelius Agrippa wrote De Occulta Philosophia in 1510 [2] Magic squares generally fall into the realm of recreational mathematics [3, 4], however a few times in the past century and more recently, they have become the interest of more-serious mathematicians.

A normal magic square is a square array of consecutive numbers from $1 \ldots n^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square of order 4 have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [5]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

NOTATIONS AND MATHEMATICAL PRELIMINARIES

Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an n^th^ order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho \qquad for \quad i = 1,2,\ldots..n \qquad (1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho \qquad for \quad i = 1,2,\ldots..n \qquad (2)$$

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$$\sum_{i=1}^{n} a_{ii} = \rho \ , \ \sum_{i=1}^{n} a_{i,n-i+1} = \rho \qquad (3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [6]

Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$. In the example given below the magic constant of A (Table:1) is 15 and B (Table:2) is 34

A =

8	1	6
3	5	7
4	9	2

Table:1

B =

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

Table:2

Strongly magic square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \ \ for \ \ i = 1,2,\ldots..n^2 \qquad (4)$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \ \ for \ \ i = 1,2,\ldots..n^2 \qquad (5)$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \ , \qquad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \qquad (6)$$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \ \ for \ \ i,j = 1,2,\ldots..n^2 \qquad (7)$$

$$where\ the\ subscripts\ are\ congruent\ modulo\ n^2$$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n^{th} order subsquare sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

PROPOSITIONS

For a Strongly Magic Square $A = [a_{ij}]$ of order 9 where $1 \le i,j \le 9$ the following properties hold

Proposition 3.1

$$a_{11} + a_{12} + a_{13} = a_{41} + a_{42} + a_{43} \quad = \quad a_{71} + a_{72} + a_{73}$$

Proof

Since for a strongly magic square A

$a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33} = \rho$ (8)

and $a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33} + a_{41} + a_{42} + a_{43} = \rho$ (9)

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Therefore equating (8) and (9) yields $a_{11}+a_{12}+a_{13}=a_{41}+a_{42}+a_{43}$
and $a_{41}+a_{42}+a_{43}=a_{71}+a_{72}+a_{73}$

Proposition 3.2

$$a_{11}+a_{13}+a_{15}+a_{21}+a_{23}+a_{25}+a_{31}+a_{33}+a_{35}=\rho$$

Proof

$$a_{11}+a_{21}+a_{31}+a_{13}+a_{23}+a_{33}=\rho-(a_{12}+a_{22}+a_{32}) \qquad (10)$$

But $a_{12}+a_{22}+a_{32}=a_{15}+a_{25}+a_{35}$ (By proposition 3.1)
Therefore (10) becomes $a_{11}+a_{21}+a_{31}+a_{13}+a_{23}+a_{33} = \rho-(a_{15}+a_{25}+a_{35})$
i.e,

$$a_{11}+a_{13}+a_{15}+a_{21}+a_{23}+a_{25}+a_{31}+a_{33}+a_{35}=\rho$$

Proposition 3.3

$$a_{11}+a_{13}+a_{15}=a_{41}+a_{43}+a_{45}=a_{71}+a_{73}+a_{75}$$

Proof

$a_{11}+a_{13}+a_{15}+a_{21}+a_{23}+a_{25}+a_{31}+a_{33}+a_{35}=\rho$ (By proposition 3.2) (11)
$Also\ a_{21}+a_{23}+a_{25}+a_{31}+a_{33}+a_{35}+a_{41}+a_{43}+a_{45}=\rho$ (By proposition 3.2) (12)
Therefore equating (11) and (12) gives $a_{11}+a_{13}+a_{15}=a_{41}+a_{43}+a_{45}$
and $a_{41}+a_{43}+a_{45}=a_{71}+a_{73}+a_{75}$

Proposition 3.4

$$a_{11}+a_{13}+a_{15}+a_{31}+a_{33}+a_{35}+a_{51}+a_{53}+a_{55}=\rho$$

Proof

$a_{11}+a_{13}+a_{15}+a_{21}+a_{23}+a_{25}+a_{31}+a_{33}+a_{35}=\rho$ (By Proposition 3.2)
But $a_{21}+a_{23}+a_{25}=a_{51}+a_{53}+a_{55}$ (By Proposition 3.3)

$$\therefore\ a_{11}+a_{13}+a_{15}+a_{31}+a_{33}+a_{35}+a_{51}+a_{53}+a_{55}=\rho$$

Proposition 3.5

$$a_{11}+a_{15}+a_{19}+a_{51}+a_{55}+a_{59}+a_{91}+a_{95}+a_{99}=\rho$$

Proof

From the definition of Strongly Magic Square

$a_{11}+a_{15}+a_{19}=\rho-(a_{12}+a_{13}+a_{14}+a_{16}+a_{17}+a_{18})$ (13)
$a_{51}+a_{55}+a_{59}=\rho-(a_{52}+a_{53}+a_{54}+a_{56}+a_{57}+a_{58})$ (14)
$a_{91}+a_{95}+a_{99}=\rho-(a_{92}+a_{93}+a_{94}+a_{96}+a_{97}+a_{98})$ (15)
Adding $(13),(14), and\ (15)$yields

$$a_{11}+a_{15}+a_{19}+a_{51}+a_{55}+a_{59}+a_{91}+a_{95}+a_{99} = 3\times\rho-(a_{12}+a_{13}+a_{14}+a_{16}+a_{17}+a_{18}+a_{52}+a_{53}+a_{54}+a_{56}+a_{57}+a_{58}+a_{92}+a_{93}+a_{94}+a_{96}+a_{97}+a_{98}) \qquad (16)$$

But $a_{52}+a_{53}+a_{54}=a_{22}+a_{23}+a_{24}$ (By Proposition 3.1)

$$a_{56}+a_{57}+a_{58}=a_{26}+a_{27}+a_{28}$$
$$a_{92}+a_{93}+a_{94}=a_{32}+a_{33}+a_{34}=a_{62}+a_{63}+a_{64}$$
$$a_{96}+a_{97}+a_{98}=a_{36}+a_{37}+a_{38}=a_{66}+a_{67}+a_{68}$$

Using these

$a_{12}+a_{13}+a_{14}+a_{52}+a_{53}+a_{54}+a_{92}+a_{93}+a_{94}=a_{12}+a_{13}+a_{14}+a_{22}+a_{23}+a_{24}+a_{32}+a_{33}+a_{34}$
Since $a_{12}+a_{13}+a_{14}+a_{22}+a_{23}+a_{24}+a_{32}+a_{33}+a_{34}=\rho$

$$a_{12}+a_{13}+a_{14}+a_{52}+a_{53}+a_{54}+a_{92}+a_{93}+a_{94}=\rho \qquad (17)$$

and $a_{16}+a_{17}+a_{18}+a_{56}+a_{57}+a_{58}+a_{96}+a_{97}+a_{98}=a_{16}+a_{17}+a_{18}+a_{26}+a_{27}+a_{28}+a_{36}+a_{37}+a_{38}$
Since $a_{16}+a_{17}+a_{18}+a_{26}+a_{27}+a_{28}+a_{36}+a_{37}+a_{38}=\rho$

$$a_{16}+a_{17}+a_{18}+a_{56}+a_{57}+a_{58}+a_{96}+a_{97}+a_{98}=\rho \qquad (18)$$

$\therefore$ (16) becomes

$$a_{11}+a_{15}+a_{19}+a_{51}+a_{55}+a_{59}+a_{91}+a_{95}+a_{99}=3\times\rho-(\rho+\rho)$$

Thus

$$a_{11}+a_{15}+a_{19}+a_{51}+a_{55}+a_{59}+a_{91}+a_{95}+a_{99}=\rho$$

Proposition 3.6

The sum of each diagonally opposite corner pairs is the same for a particular 4×4 square formed in a 9×9 strongly magic squares.(This property is called 4×4 square property)

Proof

Let c_1, c_2, c_3, c_4 be the corner elements of any 4×4 sub square of 9×9 strongly magic square then it has to be proved that

$$c_1 + c_4 = c_2 + c_3$$

Each of the corners c_1, c_2, c_3, c_4 can be considered as a corner of a 3×3 subsquare.
For example consider the 4×4 sub square $SM_4{}'$

$$SM_4{}' = \begin{bmatrix} a_{61} & a_{62} & a_{63} & a_{64} \\ a_{71} & a_{72} & a_{73} & a_{74} \\ a_{81} & a_{82} & a_{83} & a_{84} \\ a_{91} & a_{92} & a_{93} & a_{94} \end{bmatrix}$$

Here $c_1 = a_{61}$; $c_2 = a_{64}$; $c_3 = a_{91}$; $c_4 = a_{94}$
Then $c_1 = \rho - S_1$; $c_2 = \rho - S_2$; $c_3 = \rho - S_3$; $c_4 = \rho - S_4$
where $S_1 = a_{62} + a_{63} + a_{71} + a_{72} + a_{73} + a_{81} + a_{82} + a_{83}$
$S_2 = a_{62} + a_{63} + a_{72} + a_{73} + a_{74} + a_{82} + a_{83} + a_{84}$
$S_3 = a_{71} + a_{72} + a_{73} + a_{81} + a_{82} + a_{83} + a_{92} + a_{93}$
And $S_4 = a_{72} + a_{73} + a_{74} + a_{82} + a_{83} + a_{84} + a_{92} + a_{93}$
Then $c_1 + c_4 = 2 \times \rho - (S_1 + S_4)$
And $c_2 + c_3 = 2 \times \rho - (S_2 + S_3)$
But $S_1 + S_4 = a_{62} + a_{63} + a_{71} + a_{72} + a_{73} + a_{81} + a_{82} + a_{83} + a_{72} + a_{73} + a_{74} + a_{82} + a_{83} + a_{84} + a_{92} + a_{93} = S_2 + S_3$
Thus $c_1 + c_4 = c_2 + c_3$

Proposition 3.7

The sum of the three numbers along any diagonal , the numbers being separated from each other by two numbers equals $\frac{\rho}{3}$.

Proof

Let the 9×9 SMS be given to be $A = [a_{ij}]$
Consider the two main diagonals and the rows and columns containing the central element a_{55} as shown below.

a_{11}				a_{15}				a_{19}
	a_{22}			a_{25}			a_{28}	
		a_{33}		a_{35}		a_{37}		
			a_{44}	a_{45}	a_{46}			
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	a_{57}	a_{58}	a_{59}
			a_{64}	a_{65}	a_{66}			
		a_{73}		a_{75}		a_{77}		
	a_{82}			a_{85}			a_{88}	
a_{91}				a_{95}				a_{99}

Since a_{55} is common to the all these entries;
The total sum of all the numbers in the two main diagonals and the row and the column except a_{55} is given by $\sum a_{ij} = 4\rho - 4\, a_{55}$ (19)
However all these squares can be considered to be part of various squares all centred at a_{55}.
Thus
$a_{11} + a_{15} + a_{19} + a_{51} + a_{55} + a_{59} + a_{91} + a_{95} + a_{99} = \rho$ (By Proposition 3.5) (20)
$a_{33} + a_{35} + a_{37} + a_{53} + a_{55} + a_{57} + a_{73} + a_{75} + a_{77} = \rho$ (By Proposition3.4) (21)
$a_{44} + a_{45} + a_{46} + a_{54} + a_{55} + a_{56} + a_{64} + a_{65} + a_{66} = \rho$ (From definition of SMS) (22)
On adding (20) , (21) and (22) excluding a_{55} is given by
$a_{11} + a_{15} + a_{19} + a_{51} + a_{59} + a_{91} + a_{95} + a_{99} + a_{33} + a_{35} + a_{37} + a_{53} + a_{57} + a_{73} + a_{75} + a_{77} + a_{44} + a_{45} + a_{46} + a_{54} + a_{56} + a_{64} + a_{65} + a_{66} = 3\rho - 3\, a_{55}$ (23)

Using (19) and (23) ,sum of all the 9 numbers excluding a_{55} = $4\rho - 4\,a_{55} - (3\rho - 3\,a_{55})$
$= \rho - a_{55}$

Now consider the 7×7 subsquare $M_{s'}{}^{7}$

a_{22}			a_{25}			a_{28}
	a_{33}		a_{35}		a_{37}	
		a_{44}	a_{45}	a_{46}		
a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	a_{57}	a_{58}
		a_{64}	a_{65}	a_{66}		
	a_{73}		a_{75}		a_{77}	
a_{82}			a_{85}			a_{88}

Sum of all the 9 numbers including a_{55} of the 7×7 *subsquare* $M_{s'}{}^{7}$
is given by

$a_{22} + a_{25} + a_{28} + a_{52} + a_{55} + a_{58} + a_{82} + a_{85} + a_{88} = \rho$ (24)

But by 4×4 square property

$a_{22} = a_{25} + a_{52} - a_{55}$

$a_{88} = a_{58} + a_{85} - a_{55}$

Therefore $a_{22} + a_{55} + a_{88} = a_{25} + a_{52} - a_{55} + a_{58} + a_{85}$ (25)

Now (By 4×4 square property) $a_{25} + a_{58} - a_{55} = a_{28}$ (26)

Substituting (26) in (25) yields

$a_{22} + a_{55} + a_{88} = a_{52} + a_{85} + a_{28}$ (27)

and

$a_{52} + a_{85} - a_{55} = a_{82}$ (By 4×4 square property) (28)

Substituting (28) in (25) yields

$$a_{22} + a_{55} + a_{88} = a_{25} + a_{58} + a_{82}$$

Thus it follows that

$$a_{22} + a_{55} + a_{88} = a_{25} + a_{58} + a_{82} = a_{52} + a_{85} + a_{28}$$

Hence from (24) ; $3(a_{22} + a_{55} + a_{88}) = \rho$

Therefore $a_{22} + a_{55} + a_{88} = \frac{\rho}{3}$

Proposition 3.8

The sum of each diagonally opposite corner pairs is the same for a particular 7×7 square formed in a 9×9 strongly magic squares.(This property is called 7×7 square property)

Proof

Consider a 7×7 sub square in a 9×9 strongly magic square as follows

$M_s{}^{7} = [a_{ij}]$ where $i, j = 1,2 \ldots 7$

For instance let $M_s{}^{7} =$

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	a_{37}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	a_{47}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	a_{57}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	a_{67}

a_{71}	a_{72}	a_{73}	a_{74}	a_{75}	a_{76}	a_{77}

Table:3 Consider the 4×4 squares from Table:3

a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	a_{22}	a_{23}	a_{24}
a_{31}	a_{32}	a_{33}	a_{34}
a_{41}	a_{42}	a_{43}	a_{44}

$M_1{}^4$

a_{14}	a_{15}	a_{16}	a_{17}
a_{24}	a_{25}	a_{26}	a_{27}
a_{34}	a_{35}	a_{36}	a_{37}
a_{44}	a_{45}	a_{46}	a_{47}

$M_2{}^4$

a_{41}	a_{42}	a_{43}	a_{44}
a_{51}	a_{52}	a_{53}	a_{54}
a_{61}	a_{62}	a_{63}	a_{64}
a_{71}	a_{72}	a_{73}	a_{74}

$M_3{}^4$

a_{44}	a_{45}	a_{46}	a_{47}
a_{54}	a_{55}	a_{56}	a_{57}
a_{64}	a_{65}	a_{66}	a_{67}
a_{74}	a_{75}	a_{76}	a_{77}

$M_4{}^4$

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Then by 4×4 square property

$a_{11} = a_{41} + a_{14} - a_{44}$
$a_{17} = a_{14} + a_{47} - a_{44}$
$a_{71} = a_{41} + a_{74} - a_{44}$
$and\ a_{77} = a_{74} + a_{47} - a_{44}$
$a_{11} + a_{77} = a_{41} + a_{14} - a_{44} + a_{74} + a_{47} - a_{44} = a_{17} + a_{71}$

Thus $a_{11} + a_{77} = a_{17} + a_{71}$

CONCLUSION

Despite the fact that magic squares have been studied for a long time, they are still the subject of research projects. These include both mathematical-historical research, such as the discovery of unpublished magic squares of Benjamin Franklin [7], and pure mathematical research, much of which is connected with the algebraic and combinatorial geometry of polyhedra (see for example [8] and [9]).While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. Aside from mathematical research, magic squares naturally continue to be an excellent source of topics for “popular” mathematics books (see, for example, [10,11]). The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described. Physical application of magic squares is still a new topic that needs to be explored more. There are many interesting ideas for research in this field.

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